

MULTIPLE CODINGS FOR SELF-SIMILAR SETS WITH OVERLAPS

KARMA DAJANI, KAN JIANG, DERONG KONG, AND WENXIA LI

ABSTRACT. In this paper we consider a class \mathcal{E} of self-similar sets with overlaps. In particular, for a self-similar set $E \in \mathcal{E}$ and $k = 1, 2, \dots, \aleph_0$ or 2^{\aleph_0} we investigate the Hausdorff dimension of the subset $\mathcal{U}_k(E)$ which contains all points $x \in E$ having exactly k different codings. This generalizes many results obtained in [4] and [3].

1. INTRODUCTION

Non-integer base expansions were pioneered by Rényi [15] and Parry [14]. It is generally believed that a real number x has a continuum of expansions [16]. However, Erdős et al. [6] discovered that there still exist a large number of reals having exactly k different expansions, where $k = 1, 2, \dots$ or \aleph_0 . Denote by \mathcal{U}_k the set of all such reals. In particular, for $k = 1$ there are many works devoted to \mathcal{U}_1 (cf. [8, 5, 10, 12, 11]). However, when $k \geq 2$, very few is known for \mathcal{U}_k (see [17, 1, 18]).

In this paper we consider similar questions for self-similar sets with overlaps. For $1 \leq i \leq m$ let $f_i(\cdot)$ be a *similitude* on \mathbb{R} defined by

$$f_i(x) = r_i x + b_i,$$

where $r_i \in (0, 1)$ and $b_i \in \mathbb{R}$. Then there exists a unique non-empty compact set E satisfying (cf. [9, 7])

$$E = \bigcup_{i=1}^m f_i(E).$$

Date: March 31, 2016.

2010 Mathematics Subject Classification. 28A78, 28A80.

Key words and phrases. self-similar set; overlaps; multiple codings; Hausdorff dimension.

In this case, we call the couple $(E, \{f_i\}_{i=1}^m)$ a *self-similar iterated function system* (SIFS). Accordingly, the compact set E is called a *self-similar set* generated by $\{f_i\}_{i=1}^m$.

In this paper we consider a class \mathcal{E} of SIFS $(E, \{f_i\}_{i=1}^m)$ satisfying the following conditions (A)–(D). Denote by $I = [a, b]$ the convex hull of the self-similar set E . We assume that

- (A) $a = f_1(a) < f_2(a) < \cdots < f_m(a) < f_m(b) = b$.
- (B) $f_i(I) \cap f_{i+2}(I) = \emptyset$ for any $1 \leq i \leq m-2$.
- (C) There exist $i, j \in \{1, \dots, m-1\}$ such that

$$f_i(I) \cap f_{i+1}(I) = \emptyset \quad \text{and} \quad f_j(I) \cap f_{j+1}(I) \neq \emptyset.$$

- (D) If $f_i(I) \cap f_{i+1}(I) \neq \emptyset$, then there exist $u, v \geq 1$ such that

$$f_i(I) \cap f_{i+1}(I) = f_{im^u}(I) = f_{(i+1)1^v}(I),$$

$$\text{where } f_{i_1 \dots i_k}(\cdot) := f_{i_1} \circ \cdots \circ f_{i_k}(\cdot).$$

The intervals $f_i(I)$, $i = 1, \dots, m$ are called the *fundamental intervals* of $(E, \{f_i\}_{i=1}^m)$.

Then by Conditions (A)–(D) it follows that the fundamental intervals are located from left to right in the following way: the most left one is $f_1(I)$, and then the second one is $f_2(I)$, and the most right one is $f_m(I)$. Furthermore, there exist two neighbouring fundamental intervals having a non-empty intersection, and also two neighbouring fundamental intervals having an empty intersection. But any three fundamental intervals must have a null intersection. By Condition (D) it follows that a fundamental interval cannot be contained in another fundamental interval, and the intersection of fundamental intervals cannot be a singleton.

Let $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$. Then for any $x \in E$ there exists a sequence $(d_i) = d_1 d_2 \cdots \in \{1, 2, \dots, m\}^\infty$ such that (cf. [7])

$$(1.1) \quad x = \lim_{n \rightarrow \infty} f_{d_1 \dots d_n}(0) =: \pi((d_i)).$$

The sequence (d_i) is called a *coding* of x with respect to $\{f_i\}_{i=1}^m$. We point out that $x \in E$ may have multiple codings.

For $k = 1, 2, \dots, \aleph_0$ or 2^{\aleph_0} and $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$ we set

$$\mathcal{U}_k(E) := \{x \in E : x \text{ has exactly } k \text{ different codings w.r.t. } \{f_i\}_{i=1}^m\}.$$

When $k = 1$, Baker et al. [2] considered the set $\mathcal{U}_1(E)$ and gave a sufficient condition for which the underlying dynamics is a subshift of finite type. Later, Dajani and Jiang [3] considered the calculation of the Hausdorff dimension of $\mathcal{U}_1(E)$. Recently, the authors [4] considered a special candidate $(E, \{f_i\}_{i=1}^3) \in \mathcal{E}$, where

$$f_1(x) = \frac{x}{q}, \quad f_2(x) = \frac{x}{q} + 1, \quad f_3(x) = \frac{x+q}{q}$$

with $q > (3 + \sqrt{5})/2$. In particular, it was shown in [4] that the Hausdorff dimensions of $\mathcal{U}_k(E)$ are the same for all integers $k \geq 1$.

In this paper we generalize [4] and obtain the following results.

Theorem 1.1. *Let $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$. Denote by $I = [a, b]$ the convex hull of E . The following statements are equivalent.*

- (i) $f_1(I) \cap f_2(I) \neq \emptyset$ or $f_{m-1}(I) \cap f_m(I) \neq \emptyset$.
- (ii) $\dim_H \mathcal{U}_k(E) = \dim_H \mathcal{U}_1(E)$ for all integers $k \geq 1$.
- (iii) $f_1(b) \in \mathcal{U}_{\aleph_0}(E)$ or $f_m(a) \in \mathcal{U}_{\aleph_0}$.
- (iv) $|\mathcal{U}_{\aleph_0}(E)| = \aleph_0$.

Here $|A|$ denotes the cardinality of a set A .

Theorem 1.2. *Let $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$. Denote by $I = [a, b]$ the convex hull of E . The following statements are equivalent.*

- (i) $f_1(I) \cap f_2(I) = f_{m-1}(I) \cap f_m(I) = \emptyset$.
- (ii) $\dim_H \mathcal{U}_k(E) = \dim_H \mathcal{U}_1(E)$ if $k = 2^s$ with $s \in \mathbb{N}$, and $\mathcal{U}_k(E) = \emptyset$ otherwise.
- (iii) $f_1(b) \notin \mathcal{U}_{\aleph_0}$ and $f_m(a) \notin \mathcal{U}_{\aleph_0}$.
- (iv) $\mathcal{U}_{\aleph_0}(E) = \emptyset$.

These two results imply following interesting corollaries.

Corollary 1.3. *For any $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$, we have following dichotomy: either*

$$\dim_H \mathcal{U}_k(E) = \dim_H \mathcal{U}_1(E)$$

for all $k \geq 1$, or

$$\dim_H \mathcal{U}_k(E) = \dim_H \mathcal{U}_1(E)$$

if $k = 2^s$ for some $s \geq 1$, and $\mathcal{U}_k(E) = \emptyset$ otherwise.

Corollary 1.4. *For any $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$, either $|U_{\aleph_0}(E)| = \aleph_0$ or $U_{\aleph_0}(E) = \emptyset$.*

Finally, we consider the set $\mathcal{U}_{2^{\aleph_0}}(E)$ which contains all $x \in E$ having a continuum of codings.

Theorem 1.5. *Let $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$. Then $\dim_H \mathcal{U}_{2^{\aleph_0}}(E) = \dim_H E$. Furthermore,*

$$0 < \mathcal{H}^{\dim_H E}(\mathcal{U}_{2^{\aleph_0}}(E)) < \infty.$$

The rest of the paper is arranged in the following way. In Section 2 we consider the set $\mathcal{U}_k(E)$ of points having exactly k different codings, and prove the equivalences (i) \Leftrightarrow (ii) in Theorems 1.1 and 1.2, respectively. In Section 3 we investigate the set $\mathcal{U}_{\aleph_0}(E)$ which contains all $x \in E$ having countably infinitely many codings, and finish the proofs of Theorems 1.1 and 1.2. The proof of Theorem 1.5 for the set of points having a continuum of codings will be presented in Section 4. Finally, in Section 5 we consider some examples.

2. FINITE CODINGS

Let $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$. In this section we will consider the set $\mathcal{U}_k(E)$ which contains all $x \in E$ having exactly k different codings with respect to $\{f_i\}_{i=1}^m$, and prove the equivalences (i) \Leftrightarrow (ii) in Theorems 1.1 and 1.2, respectively.

First we give some properties of the SIFS $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$.

Lemma 2.1. *Let $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$. If $f_{im^u}(I) = f_{(i+1)1^v}(I)$ for some $1 \leq i \leq m-1$ and $u, v \geq 1$, then $f_{im^u}(\cdot) = f_{(i+1)1^v}(\cdot)$.*

Proof. Note that for any $x \in \mathbb{R}$ we can write

$$(2.1) \quad f_{im^u}(x) = rx + t, \quad f_{(i+1)1^v}(x) = r'x + t',$$

for some $r, r' \in (0, 1)$ and $t, t' \in \mathbb{R}$. Suppose that $I = [a, b]$. Then by using $f_{im^u}(I) = f_{(i+1)1^v}(I)$ it follows that

$$ra + t = f_{im^u}(a) = f_{(i+1)1^v}(a) = r'a + t',$$

$$rb + t = f_{im^u}(b) = f_{(i+1)1^v}(b) = r'b + t'.$$

This implies $r = r'$ and $t = t'$. By (2.1) we have $f_{im^u}(\cdot) = f_{(i+1)1^v}(\cdot)$. \square

Lemma 2.2. *Let $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$. Suppose*

$$x \in f_i(I) \cap f_{i+1}(I) = f_{im^u}(I) = f_{(i+1)1^v}(I)$$

for some $i \in \{1, \dots, m-1\}$ and $u, v \geq 1$. Then all codings of x either begin with im^{u-1} or with $(i+1)1^{v-1}$.

Proof. Let (d_i) be a coding of x w.r.t. $\{f_i\}_{i=1}^m$. Note that $x = \pi((d_i)) \in f_i(I) \cap f_{i+1}(I)$ and that any three fundamental intervals have an empty intersection. Then

$$d_1 = i \quad \text{or} \quad i+1.$$

If $d_1 = i$ with $u = 1$ or $d_1 = i+1$ with $v = 1$, then we are done. So, we will finish the proof by considering the following two cases.

Case (I). $d_1 = i$ and $u > 1$. Note that $x = \pi(id_2d_3\cdots) \in f_{im^u}(I)$. Then

$$(2.2) \quad \pi(d_2d_3\cdots) \in f_{m^u}(I),$$

and we claim that $d_2 = m$.

Suppose on the contrary that $d_2 \neq m$. Then by the location of these fundamental intervals we have $d_2 = m-1$. So, by (2.2) and Condition (D) it follows that

$$\pi(d_2d_3\cdots) \in f_{m^u}(I) \cap (f_{m-1}(I) \cap f_m(I)) \subseteq f_{m^u}(I) \cap f_{m1}(I),$$

leading to a contradiction since $f_1(I) \cap f_m(I) = \emptyset$.

Therefore, $d_2 = m$ and $u > 1$. By iteration it follows that $d_2 \cdots d_u = m^{u-1}$.

Case (II). $d_1 = i+1$. Note that $x = \pi((i+1)d_2d_3\cdots) \in f_{(i+1)1^v}(I)$. Then

$$(2.3) \quad \pi(d_2d_3\cdots) \in f_{1^v}(I),$$

and we claim that $d_2 = 1$.

Suppose on the contrary that $d_2 \neq 1$. Then $d_2 = 2$, and therefore by (2.3) and Condition (D) it follows that

$$\pi(d_2d_3\cdots) \in f_{1^v}(I) \cap (f_1(I) \cap f_2(I)) \subseteq f_{1^v}(I) \cap f_{1m}(I),$$

leading to a contradiction with $f_1(I) \cap f_m(I) = \emptyset$.

Therefore, $d_2 = 1$. By iteration we conclude that $d_2 \cdots d_v = 1^{v-1}$. \square

The upper bound of $\dim_H \mathcal{U}_k(E)$ can be deduced directly.

Lemma 2.3. *Let $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$. Then for any $k \geq 1$ we have*

$$\dim_H \mathcal{U}_k(E) \leq \dim_H \mathcal{U}_1(E).$$

Proof. Take $x \in \mathcal{U}_k(E)$. Then all of its codings eventually belong to $\mathcal{U}'_1(E) := \{(c_i) \in \{1, \dots, m\}^\infty : \pi((c_i)) \in \mathcal{U}_1(E)\}$. Therefore, the lemma follows by observing that

$$\mathcal{U}_k(E) \subseteq \bigcup_{n=1}^{\infty} \bigcup_{d_1 \cdots d_n \in \{1, 2, \dots, m\}^n} f_{d_1 \cdots d_n}(\mathcal{U}_1(E)).$$

\square

For the lower bound of $\dim_H \mathcal{U}_k(E)$ we split the proof into the following four subsections.

- $f_1(I) \cap f_2(I) \neq \emptyset$ but $f_{m-1}(I) \cap f_m(I) = \emptyset$;
- $f_1(I) \cap f_2(I) = \emptyset$ but $f_{m-1}(I) \cap f_m(I) \neq \emptyset$;
- $f_1(I) \cap f_2(I) \neq \emptyset$ and $f_{m-1}(I) \cap f_m(I) \neq \emptyset$;
- $f_1(I) \cap f_2(I) = f_{m-1}(I) \cap f_m(I) = \emptyset$.

2.1. $f_1(I) \cap f_2(I) \neq \emptyset$ **but** $f_{m-1}(I) \cap f_m(I) = \emptyset$. In the following lemma we will show that the set of $x \in \mathcal{U}_1(E)$ with its coding starting at $m-1$ has the same Hausdorff dimension as $\mathcal{U}_1(E)$.

Lemma 2.4. *Let $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$. If $f_{m-1}(I) \cap f_m(I) = \emptyset$, then*

$$\dim_H f_{m-1}(E) \cap \mathcal{U}_1(E) = \dim_H \mathcal{U}_1(E),$$

Proof. Let

$$\begin{aligned} \phi : \mathcal{U}_1(E) &\longrightarrow f_{(m-1)m}(E) \cap \mathcal{U}_1(E) \\ x &\longmapsto f_{(m-1)m}(x). \end{aligned}$$

First we prove that ϕ is well-defined. Take $x \in \mathcal{U}_1(E)$. It suffices to prove $f_{(m-1)m}(x) \in \mathcal{U}_1(E)$.

Note that $f_{m-1}(I) \cap f_m(I) = \emptyset$. Then the locations of the fundamental intervals yield that

$$(2.4) \quad f_i(I) \cap f_m(I) = \emptyset \quad \text{for any } i \neq m.$$

So, $f_m(x) \in \mathcal{U}_1(E)$. Suppose on the contrary that $f_{(m-1)m}(x) \notin \mathcal{U}_1(E)$. Then by (2.4) and Condition (D) it follows that

$$f_{(m-1)m}(x) \in f_{m-2}(I) \cap f_{m-1}(I) \subseteq f_{(m-1)1}(I).$$

This implies $f_m(x) \in f_1(I)$, leading to a contradiction with (2.4).

Therefore, ϕ is well-defined. Note that ϕ is a similitude. Then one can easily check that ϕ is bijective. In particular, ϕ is a bi-Lipschitz map between $\mathcal{U}_1(E)$ and $f_{(m-1)m}(E) \cap \mathcal{U}_1(E)$. Hence,

$$\begin{aligned} \dim_H \mathcal{U}_1(E) &= \dim_H f_{(m-1)m}(E) \cap \mathcal{U}_1(E) \leq \dim_H f_{m-1}(E) \cap \mathcal{U}_1(E) \\ &\leq \dim_H \mathcal{U}_1(E). \end{aligned}$$

□

Now we consider the lower bound of $\dim_H \mathcal{U}_k(E)$.

Lemma 2.5. *Let $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$. If $f_1(I) \cap f_2(I) \neq \emptyset$ but $f_{m-1}(I) \cap f_m(I) = \emptyset$, then for any $k \geq 1$ we have*

$$\dim_H \mathcal{U}_k(E) \geq \dim_H \mathcal{U}_1(E).$$

Proof. Note that $f_1(I) \cap f_2(I) \neq \emptyset$. Then by Condition (D) there exist $u, v \geq 1$ such that

$$f_1(I) \cap f_2(I) = f_{1m^u}(I) = f_{21^v}(I).$$

By Lemma 2.1 we have $f_{1m^u}(\cdot) = f_{21^v}(\cdot)$.

Take $\pi(\mathbf{c}) := \pi((c_i)) \in f_{m-1}(E) \cap \mathcal{U}_1(E)$. Then $c_1 = m - 1$. Now we claim that

$$x_s := \pi(1m^{us}\mathbf{c})$$

has exactly $s + 1$ different codings. We will prove this by induction on s .

Suppose $s = 0$. Then $x_0 = \pi(1\mathbf{c})$. Denote by $I = [a, b]$. Then by (2.4) and Condition (A) it follows that

$$x_0 = \pi(1(m-1)c_2c_3\cdots) \leq f_{1(m-1)}(b) < f_{1m^u}(a) = f_{21^v}(a) = f_2(a).$$

This together with $\pi(\mathbf{c}) = \pi((m-1)c_2c_3\cdots) \in \mathcal{U}_1(E)$ implies that x_0 has a unique coding.

Now suppose that x_s has $s+1$ different codings for some $s \geq 0$. We will prove that x_{s+1} has exactly $s+2$ codings. Note that

$$(2.5) \quad x_{s+1} = f_{1m^u}(\pi(m^{us}\mathbf{c})) = f_{21^v}(\pi(m^{us}\mathbf{c})) = f_{21^{v-1}}(x_s).$$

By the induction hypothesis this implies that x_{s+1} has at least $s+2$ different codings: one is $1m^{u(s+1)}\mathbf{c}$, and the others start at 21^{v-1} .

In the following we will prove that x_{s+1} has exactly $s+2$ codings. Suppose that (d_i) is a coding of x_{s+1} . By (2.5) and Lemma 2.2 it follows that

$$d_1 \cdots d_u = 1m^{u-1} \quad \text{or} \quad d_1 \cdots d_v = 21^{v-1}.$$

So, it suffices to prove that $d_1 \cdots d_u = 1m^{u-1}$ implies $(d_i) = 1m^{u(s+1)}\mathbf{c}$.

Suppose $d_1 \cdots d_u = 1m^{u-1}$. Then by (2.5) it gives

$$\pi(d_{u+1}d_{u+2}\cdots) = f_{m^{us+1}}(\pi(\mathbf{c})).$$

By (2.4) it follows that

$$d_{u+1} \cdots d_{u(s+1)+1} = m^{us+1}, \quad \pi(d_{u(s+1)+2}d_{u(s+1)+3}\cdots) = \pi(\mathbf{c}).$$

Observe that $\pi(\mathbf{c}) \in \mathcal{U}_1(E)$. Then $(d_i) = 1m^{u(s+1)}\mathbf{c}$.

Hence, we conclude by induction that x_s has exactly $s+1$ codings for any integers $s \geq 0$. Note that $\pi(\mathbf{c})$ is taken from $f_{m-1}(E) \cap \mathcal{U}_1(E)$ arbitrarily. Then

$$\{x_s = \pi(1m^{us}\mathbf{c}) : \pi(\mathbf{c}) \in f_{m-1}(E) \cap \mathcal{U}_1(E)\} \subseteq \mathcal{U}_{s+1}(E).$$

By Lemma 2.4 it follows that for any integers $s \geq 0$ we have

$$\dim_H \mathcal{U}_{s+1}(E) \geq \dim_H f_{m-1}(E) \cap \mathcal{U}_1(E) = \dim_H \mathcal{U}_1(E).$$

□

2.2. $f_1(I) \cap f_2(I) = \emptyset$ but $f_{m-1}(I) \cap f_m(I) \neq \emptyset$. First we show that the set of $x \in \mathcal{U}_1(E)$ with its coding beginning with 2 has the same Hausdorff dimension as $\mathcal{U}_1(E)$.

Lemma 2.6. *Let $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$. If $f_1(I) \cap f_2(I) = \emptyset$, then*

$$\dim_H f_2(E) \cap \mathcal{U}_1(E) = \dim_H \mathcal{U}_1(E),$$

Proof. In a similar way as in Lemma 2.4 one can show that the following map

$$\begin{aligned} \psi : \quad \mathcal{U}_1(E) &\longrightarrow f_{21}(E) \cap \mathcal{U}_1(E) \\ x &\longmapsto f_{21}(x) \end{aligned}$$

is bi-Lipschitz. Then

$$\begin{aligned} \dim_H \mathcal{U}_1(E) &= \dim_H f_{21}(E) \cap \mathcal{U}_1(E) \leq \dim_H f_2(E) \cap \mathcal{U}_1(E) \\ &\leq \dim_H \mathcal{U}_1(E). \end{aligned}$$

□

Lemma 2.7. *Let $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$. If $f_1(I) \cap f_2(I) = \emptyset$ but $f_{m-1}(I) \cap f_m(I) \neq \emptyset$, then for any $k \geq 1$ we have*

$$\dim_H \mathcal{U}_k(E) \geq \dim_H \mathcal{U}_1(E).$$

Proof. Note that $f_{m-1}(I) \cap f_m(I) \neq \emptyset$. Then by Condition (D) there exist $u, v \geq 1$ such that

$$f_{m-1}(I) \cap f_m(I) = f_{(m-1)m^u}(I) = f_{m1^v}(I).$$

By Lemma 2.1 we have $f_{(m-1)m^u}(\cdot) = f_{m1^v}(\cdot)$.

Take $\pi(\mathbf{c}) = \pi(c_1 c_2 \cdots) \in f_2(E) \cap \mathcal{U}_1(E)$. Then $c_1 = 2$. For $s \geq 0$ we define

$$y_s := \pi(m1^{vs}\mathbf{c}).$$

In a similar way as in the proof of Lemma 2.5 one can prove that y_s has exactly $s + 1$ different codings.

Note that $\pi(\mathbf{c})$ is taken from $f_2(E) \cap \mathcal{U}_1(E)$ arbitrarily. Then by Lemma 2.6 it follows that for any $s \geq 0$ we have

$$\begin{aligned} \dim_H \mathcal{U}_{s+1}(E) &\geq \dim_H \{y_s = \pi(m1^{vs}\mathbf{c}) : \pi(\mathbf{c}) \in f_2(E) \cap \mathcal{U}_1(E)\} \\ &= \dim_H f_2(E) \cap \mathcal{U}_1(E) = \dim_H \mathcal{U}_1(E). \end{aligned}$$

□

2.3. $f_1(I) \cap f_2(I) \neq \emptyset$ **and** $f_{m-1}(I) \cap f_m(I) \neq \emptyset$. By Condition (C) we may assume that $f_i(I) \cap f_{i+1}(I) = \emptyset$ for some $i \in \{2, \dots, m-2\}$. In the following lemma we will show that the Hausdorff dimension of $\mathcal{U}_1(E)$ is dominated by the subset which contains all $x \in \mathcal{U}_1(E)$ with its coding starting at i or $i+1$.

Lemma 2.8. *Let $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$. If $f_i(I) \cap f_{i+1}(I) = \emptyset$ for some $i \in \{2, \dots, m-2\}$, then*

$$\dim_H \mathcal{U}_1(E) = \dim_H f_i(E) \cap \mathcal{U}_1(E) = \dim_H f_{i+1}(E) \cap \mathcal{U}_1(E).$$

Proof. Note that $\mathcal{U}_1(E) = \bigcup_{j=1}^m f_j(E) \cap \mathcal{U}_1(E)$. It suffices to prove

$$(2.6) \quad \dim_H f_i(E) \cap \mathcal{U}_1(E) \geq \dim_H \bigcup_{j=i+1}^m f_j(E) \cap \mathcal{U}_1(E)$$

and

$$\dim_H f_{i+1}(E) \cap \mathcal{U}_1(E) \geq \dim_H \bigcup_{j=1}^i f_j(E) \cap \mathcal{U}_1(E)$$

Without loss of generality we only prove (2.6). Let

$$\begin{aligned} \varphi : \bigcup_{j=i+1}^m f_j(E) \cap \mathcal{U}_1(E) &\longrightarrow f_i(E) \cap \mathcal{U}_1(E) \\ x &\mapsto f_i(x). \end{aligned}$$

First we prove that φ is well-defined. Take $x \in \bigcup_{j=i+1}^m f_j(E) \cap \mathcal{U}_1(E)$. It suffices to prove that $f_i(x) \in \mathcal{U}_1(E)$. Suppose on the contrary that $f_i(x) \notin \mathcal{U}_1(E)$. Note that $f_i(I) \cap f_{i+1}(I) = \emptyset$. Then by the locations of the fundamental intervals it follows that

$$f_i(x) \in f_{i-1}(I) \cap f_i(I) \subseteq f_{i1}(I).$$

This implies that $x \in f_1(I)$, leading to contradiction since $f_1(I) \cap \bigcup_{j=i+1}^m f_j(I) = \emptyset$.

Therefore, φ is well-defined. Note that φ is a similitude. Hence,

$$\begin{aligned} \dim_H f_i(E) \cap \mathcal{U}_1(E) &\geq \dim_H \varphi \left(\bigcup_{j=i+1}^m f_j(E) \cap \mathcal{U}_1(E) \right) \\ &= \dim_H \bigcup_{j=i+1}^m f_j(E) \cap \mathcal{U}_1(E). \end{aligned}$$

This establishes (2.6). \square

Lemma 2.9. *Let $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$. If $f_1(I) \cap f_2(I) \neq \emptyset$ and $f_{m-1}(I) \cap f_m(I) \neq \emptyset$, then for any $k \geq 1$ we have*

$$\dim_H \mathcal{U}_k(E) \geq \dim_H \mathcal{U}_1(E).$$

Proof. Suppose $f_i(I) \cap f_{i+1}(I) = \emptyset$ for some $i \in \{2, \dots, m-2\}$. Then by Lemma 2.8 it follows that

$$(2.7) \quad \dim_H \mathcal{U}_1(E) = \dim_H f_i(E) \cap \mathcal{U}_1(E).$$

Note that $f_1(I) \cap f_2(I) \neq \emptyset$. Then by Condition (D) there exist $u, v \geq 1$ such that

$$f_1(I) \cap f_2(I) = f_{1m^u}(I) = f_{21^v}(I).$$

By Lemma 2.1 we have $f_{1m^u}(\cdot) = f_{21^v}(\cdot)$.

Take $\pi(\mathbf{c}) = \pi((c_i)) \in f_i(E) \cap \mathcal{U}_1(E)$. Then $c_1 = i$. Now we claim that

$$z_s := \pi(1m^{us}\mathbf{c})$$

has exactly $s+1$ different codings. We will prove this by induction on s .

Suppose $s = 0$. Then $z_0 = \pi(1\mathbf{c})$. Note that $f_i(I) \cap f_{i+1}(I) = \emptyset$ for some $i \in \{2, \dots, m-2\}$. Denote by $I = [a, b]$. Then by Condition (A) it follows that

$$z_0 = \pi(1ic_2c_3\cdots) \leq f_{1i}(b) < f_{1m^u}(a) = f_{21^v}(a) = f_2(a).$$

This together with $\pi(\mathbf{c}) = \pi(ic_2c_3\cdots) \in \mathcal{U}_1(E)$ implies that z_0 has a unique coding.

Now suppose that z_s has $s+1$ different codings for some $s \geq 0$. We will prove that z_{s+1} has exactly $s+2$ codings. Note that

$$(2.8) \quad z_{s+1} = f_{1m^u}(\pi(m^{us}\mathbf{c})) = f_{21^v}(\pi(m^{us}\mathbf{c})) = f_{21^{v-1}}(z_s).$$

By the induction hypothesis this implies that z_{s+1} has at least $s+2$ different codings: one is $1m^{u(s+1)}\mathbf{c}$, and the others start at 21^{v-1} .

In the following we will prove that z_{s+1} has exactly $s+2$ codings. Suppose that (d_i) is a coding of z_{s+1} . Then by (2.8) and Lemma 2.2 it

follows that

$$d_1 \cdots d_u = 1m^{u-1} \quad \text{or} \quad d_1 \cdots d_v = 21^{v-1}.$$

So, it suffices to prove that $d_1 \cdots d_u = 1m^{u-1}$ implies $(d_i) = 1m^{u(s+1)}\mathbf{c}$.

Suppose $d_1 \cdots d_u = 1m^{u-1}$. We claim that $d_{u+1} \cdots d_{u(s+1)+1} = m^{us+1}$. Let $1 \leq j \leq us+1$ be the smallest integer such that $d_{u+j} \neq m$. Then by (2.8) we have

$$(2.9) \quad \pi(d_{u+j}d_{u+j+1} \cdots) = f_{m^{us+2-j}}(\pi(\mathbf{c})).$$

Then by the locations of the fundamental intervals we have $d_{u+j} = m-1$. Therefore, by (2.9) and Condition (D) it follows that

$$f_{m^{us+2-j}}(\pi(\mathbf{c})) \in f_{m-1}(I) \cap f_m(I) \subseteq f_{m1}(I).$$

This implies that $f_{m^{us+1-j}}(\pi(\mathbf{c})) \in f_1(I)$. When $j < us+1$ we obtain $f_m(I) \cap f_1(I) \neq \emptyset$, leading to a contradiction since $f_i(I) \cap f_{i+1}(I) = \emptyset$. When $j = us+1$ we get $\pi(\mathbf{c}) \in f_1(I)$, leading to a contradiction since $\pi(\mathbf{c}) = \pi(ic_2c_3 \cdots) \in \mathcal{U}_1(E)$. Thus, $(d_i) = 1m^{u(s+1)}\mathbf{c}$.

Hence, we conclude by induction that z_s has exactly $s+1$ codings for any $s \geq 0$. Note that $\pi(\mathbf{c})$ is taken from $f_i(E) \cap \mathcal{U}_1(E)$ arbitrarily. Then

$$\{z_s = \pi(1m^{us}\mathbf{c}) : \pi(\mathbf{c}) \in f_i(E) \cap \mathcal{U}_1(E)\} \subseteq \mathcal{U}_{s+1}(E).$$

Hence, by (2.7) we have for any $s \geq 0$ that

$$\dim_H \mathcal{U}_{s+1}(E) \geq \dim_H f_i(E) \cap \mathcal{U}_1(E) = \dim_H \mathcal{U}_1(E).$$

□

2.4. $f_1(I) \cap f_2(I) = f_{m-1}(I) \cap f_m(I) = \emptyset$. Let $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$. First we show that $\mathcal{U}_k(E)$ is empty if $k \neq 2^s$ with $s \in \mathbb{N}$.

Lemma 2.10. *Let $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$. If*

$$f_1(I) \cap f_2(I) = f_{m-1}(I) \cap f_m(I) = \emptyset,$$

then $\mathcal{U}_k(E) = \emptyset$ for any $k \neq 2^s$ with $s \in \mathbb{N}$.

Proof. For $x \in E$ we denote by $N(x)$ the number of different codings of x with respect to $\{f_i\}_{i=1}^m$. Let $k \geq 2$ and take $x \in \mathcal{U}_k(E)$. Then $N(x) = k$. So, there exist $i \in \{2, \dots, m-2\}$ and

$$(2.10) \quad x_1 \in \mathcal{U}_k(E) \cap f_i(I) \cap f_{i+1}(I)$$

such that $N(x) = N(x_1)$.

Note that $f_i(I) \cap f_{i+1}(I) \neq \emptyset$. Then by Condition (D) there exist $u, v \geq 1$ such that

$$(2.11) \quad f_i(I) \cap f_{i+1}(I) = f_{im^u}(I) = f_{(i+1)1^v}(I).$$

Observe that $f_1(I) \cap f_2(I) = f_{m-1}(I) \cap f_m(I) = \emptyset$. Then by the locations of the fundamental intervals we obtain

$$(2.12) \quad f_1(I) \cap f_i(I) = \emptyset, \quad f_j(I) \cap f_m(I) = \emptyset$$

for any $i \neq 1$ and any $j \neq m$.

Therefore, by (2.10)–(2.12) it follows that all codings of x_1 either start with im^u or with $(i+1)1^v$. Note by using (2.11) in Lemma 2.1 that $f_{im^u}(\cdot) = f_{(i+1)1^v}(\cdot)$. Then there exists $y \in E$ such that $x_1 = f_{im^u}(y) = f_{(i+1)1^v}(y)$. Furthermore,

$$N(x_1) = N(f_{m^u}(y)) + N(f_{1^v}(y)) = 2N(y),$$

where the last equality holds by (2.12) that

$$N(f_{m^u}(y)) = N(y) = N(f_{1^v}(y)).$$

Hence, we conclude that $N(x) = N(x_1) = 2N(y)$.

By iteration it follows that $N(x)$ must be of the form 2^s for some $s \geq 1$. This completes the proof. \square

Lemma 2.11. *Let $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$. If*

$$f_1(I) \cap f_2(I) = f_{m-1}(I) \cap f_m(I) = \emptyset,$$

then $\dim_H \mathcal{U}_{2^s}(E) \geq \dim_H \mathcal{U}_1(E)$ for any $s \in \mathbb{N}$.

Proof. We will prove this lemma by induction on s . Clearly, the lemma follows if $s = 0$.

Now we assume that $\dim_H \mathcal{U}_{2^s}(E) \geq \dim_H \mathcal{U}_1(E)$ for some $s \geq 0$. In the following it suffices to prove that

$$\dim_H \mathcal{U}_{2^{s+1}}(E) \geq \dim_H \mathcal{U}_{2^s}(E).$$

By Condition (C) there exists $i \in \{2, \dots, m-2\}$ for which $f_i(I) \cap f_{i+1}(I) \neq \emptyset$. Then by Condition (D) we can find $u, v \geq 1$ such that

$$f_i(I) \cap f_{i+1}(I) = f_{im^u}(I) = f_{(i+1)1^v}(I).$$

By Lemma 2.1 this yields that $f_{im^u}(\cdot) = f_{(i+1)1^v}(\cdot)$. Note that any three fundamental intervals have an empty intersection. So, by (2.12) it follows that

$$\{f_{im^u}(x) = f_{(i+1)1^v}(x) : x \in \mathcal{U}_{2^s}(E)\} \subseteq \mathcal{U}_{2^{s+1}}(E).$$

Hence, $\dim_H \mathcal{U}_{2^{s+1}}(E) \geq \dim_H \mathcal{U}_{2^s}(E)$.

By induction we conclude that $\dim_H \mathcal{U}_{2^s}(E) \geq \dim_H \mathcal{U}_1(E)$ for any $s \geq 0$. \square

Now we give the proof of the equivalence of (i) and (ii) in Theorems 1.1 and 1.2.

Proof of Theorems 1.1 and 1.2 (i) \Leftrightarrow (ii). By Lemmas 2.3, 2.5, 2.7 and 2.9 it follows that if $f_1(I) \cap f_2(I) \neq \emptyset$ or $f_{m-1}(I) \cap f_m(I) \neq \emptyset$, then

$$\dim_H \mathcal{U}_k(E) = \dim_H \mathcal{U}_1(E) \quad \text{for all } k \geq 1.$$

On the other hand, if $f_1(I) \cap f_2(I) = f_{m-1}(I) \cap f_m(I) = \emptyset$, then by Lemmas 2.3, 2.10 and 2.11 it follows that

$$\begin{cases} \dim_H \mathcal{U}_k(E) = \dim_H \mathcal{U}_1(E) & \text{if } k = 2^s, \\ \mathcal{U}_k(E) = \emptyset & \text{otherwise.} \end{cases}$$

This completes the proof. \square

3. COUNTABLE CODINGS

In this section we will consider the set $\mathcal{U}_{\aleph_0}(E)$, and prove the equivalences (i) \Leftrightarrow (iii) \Leftrightarrow (iv) in Theorems 1.1 and 1.2, respectively. First we prove the equivalence (i) \Leftrightarrow (iii).

Lemma 3.1. *Let $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$. Denote by $I = [a, b]$ the convex hull of E .*

- (A) $f_1(I) \cap f_2(I) \neq \emptyset$ if and only if $f_1(b) \in \mathcal{U}_{\aleph_0}(E)$.
 (B) $f_{m-1}(I) \cap f_m(I) \neq \emptyset$ if and only if $f_m(a) \in \mathcal{U}_{\aleph_0}(E)$.

Proof. Since the proofs of (A) and (B) are similar, we only prove (A).

First we consider the sufficiency. Suppose on the contrary that $f_1(I) \cap f_2(I) = \emptyset$. Then by the locations of the fundamental intervals we have

$$f_1(I) \cap f_i(I) = \emptyset \quad \text{for any } i \neq 1.$$

Note that $b = \pi(m^\infty) \in \mathcal{U}_1(E)$. Then $f_1(b) \in \mathcal{U}_1(E)$.

Now we turn to prove the necessity. Suppose $f_1(I) \cap f_2(I) \neq \emptyset$. Then by Condition (D) there exist $u, v \geq 1$ such that

$$f_1(I) \cap f_2(I) = f_{1m^u}(I) = f_{21^v}(I).$$

By Lemma 2.1 it gives that $f_{1m^u}(\cdot) = f_{21^v}(\cdot)$. Then

$$(3.1) \quad f_1(b) = \pi(1m^\infty) = \pi(21^{v-1}1m^\infty) = \cdots = \pi((21^{v-1})^s 1m^\infty) = \cdots.$$

This implies that $f_1(b)$ has at least countably infinitely many codings.

In the following we show that $f_1(b)$ indeed has countably infinitely many codings. Suppose that (d_i) is a coding of $f_1(b)$. By (3.1) and Lemma 2.2 it follows that $d_1 \cdots d_u = 1m^{u-1}$ or $d_1 \cdots d_v = 21^{v-1}$.

- If $d_1 \cdots d_u = 1m^{u-1}$, then by (3.1) we have

$$\pi(d_{u+1}d_{u+2} \cdots) = \pi(m^\infty) \in \mathcal{U}_1(E).$$

This implies that $(d_i) = 1m^\infty$.

- If $d_1 \cdots d_v = 21^{v-1}$, then by (3.1) it yields that

$$\pi(d_{v+1}d_{v+2} \cdots) = \pi(1m^\infty) = f_1(b).$$

By iteration of the above arguments it follows that all the codings of $f_1(b)$ are of the form

$$(21^{v-1})^s 1m^\infty, \quad s \geq 0.$$

Hence, $f_1(b) \in \mathcal{U}_{\aleph_0}(E)$. This establishes the lemma. \square

Proof of Theorems 1.1 and 1.2 (i) \Leftrightarrow (iii). The equivalences of (i) and (iii) in Theorems 1.1 and 1.2 follows directly by Lemma 3.1. \square

Lemma 3.2. *Let $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$. If $f_1(I) \cap f_2(I) = f_{m-1}(I) \cap f_m(I) = \emptyset$, then $\mathcal{U}_{\aleph_0}(E) = \emptyset$.*

Proof. Suppose on the contrary that $\mathcal{U}_{\aleph_0}(E) \neq \emptyset$. Take $x \in \mathcal{U}_{\aleph_0}(E)$. Then x must have a coding (d_i) satisfying

$$(3.2) \quad x_n := \pi(d_{n+1}d_{n+2}\cdots) \in E \cap \bigcup_{i=2}^{m-2} f_i(I) \cap f_{i+1}(I)$$

for infinitely many $n \geq 1$.

Take n satisfying (3.2) and assume that

$$x_n \in E \cap f_i(I) \cap f_{i+1}(I)$$

for some $2 \leq i \leq m-2$. By Condition (D) there exist $u, v \geq 1$ such that

$$(3.3) \quad x_n \in f_i(I) \cap f_{i+1}(I) = f_{im^u}(I) = f_{(i+1)1^v}(I).$$

Note that $f_1(I) \cap f_2(I) = f_{m-1}(I) \cap f_m(I) = \emptyset$. Then by the locations of the fundamental intervals it follows that

$$(3.4) \quad f_1(I) \cap f_j(I) = f_\ell(I) \cap f_m(I) = \emptyset$$

for any $j \neq 1$ and any $\ell \neq m$. Therefore, by (3.3) and (3.4) it follows that

$$d_{n+1}\cdots d_{n+u+1} = im^u \quad \text{or} \quad d_{n+1}\cdots d_{n+v+1} = (i+1)1^v.$$

Note by using (3.3) in Lemma 2.1 we have $f_{im^u}(\cdot) = f_{(i+1)1^v}(\cdot)$. Therefore, we have a substitution in $d_{n+1}d_{n+2}\cdots$ by considering $im^u \sim (i+1)1^v$.

In terms of (3.2) and by iteration it follows that there exist infinitely many independent substitutions in (d_i) . This implies that x has a continuum of codings, leading to a contradiction with $x \in \mathcal{U}_{\aleph_0}(E)$. \square

Lemma 3.3. *Let $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$. If $f_1(I) \cap f_2(I) \neq \emptyset$ or $f_{m-1}(I) \cap f_m(I) \neq \emptyset$, then $|\mathcal{U}_{\aleph_0}(E)| = \aleph_0$.*

Proof. Without loss of generality we may assume $f_1(I) \cap f_2(I) \neq \emptyset$ but $f_{m-1}(I) \cap f_m(I) = \emptyset$. By Condition (D) there exist $u, v \geq 1$ such that

$$(3.5) \quad f_1(I) \cap f_2(I) = f_{1m^u}(I) = f_{21^v}(I).$$

Then by Lemma 2.1 it yields that $f_{1m^u}(\cdot) = f_{21^v}(\cdot)$.

Denote by $I = [a, b]$. First we claim that $f_{1^n}(b) \in \mathcal{U}_{\aleph_0}(E)$ for any $n \geq 1$. We will prove this by induction on n . Clearly, for $n = 1$ we have by Lemma 3.1 that $f_1(b) \in \mathcal{U}_{\aleph_0}(E)$.

Suppose that $f_{1^n}(b) \in \mathcal{U}_{\aleph_0}(E)$ for some $n \geq 1$. Now we consider $f_{1^{n+1}}(b)$. If $f_{1^{n+1}}(b) \in f_2(I)$, then by Condition (D) it follows that

$$f_{1^{n+1}}(b) \in f_1(I) \cap f_2(I) \subseteq f_{1m}(I).$$

This implies $f_{1^n}(b) \in f_m(I)$, leading to a contradiction since $f_1(I) \cap f_m(I) = \emptyset$.

Therefore, we conclude by induction that $\{f_{1^n}(b) : n \geq 1\} \subseteq \mathcal{U}_{\aleph_0}(E)$. In the following it suffices to prove that any $x \in \mathcal{U}_{\aleph_0}(E)$ must have a coding ending with $1m^\infty \sim (21^{v-1})^\infty$.

Take $x \in \mathcal{U}_{\aleph_0}(E)$. Suppose on the contrary that all the codings of x do not end with $1m^\infty \sim (21^{v-1})^\infty$. Note that x has a coding (d_i) satisfying

$$(3.6) \quad \pi(d_{n+1}d_{n+2}\cdots) \in E \cap \bigcup_{i=1}^{m-2} f_i(I) \cap f_{i+1}(I)$$

for infinitely many $n \geq 1$.

Fix n satisfying (3.6), and assume that $\pi(d_{n+1}d_{n+2}\cdots) \in f_i(I) \cap f_{i+1}(I)$ for some $i \in \{1, \dots, m-2\}$. Then by Condition (D) there exist $p, q \geq 1$ such that

$$(3.7) \quad \pi(d_{n+1}d_{n+2}\cdots) \in f_i(I) \cap f_{i+1}(I) = f_{im^p}(I) = f_{(i+1)1^q}(I).$$

By Lemmas 2.1 and 2.2 it follows that $f_{im^p}(\cdot) = f_{(i+1)1^q}(\cdot)$, and

$$d_{n+1}\cdots d_{n+p} = im^{p-1} \quad \text{or} \quad d_{n+1}\cdots d_{n+q} = (i+1)1^{q-1}.$$

Now we split the proof into the following two cases.

Case (I). $d_{n+1}\cdots d_{n+p} = im^{p-1}$. Then by (3.7) and using $f_{m-1}(I) \cap f_m(I) = \emptyset$ it follows that $d_{n+p+1} = m$. Therefore, we have a substitution by replacing $d_{n+1}\cdots d_{n+p+1} = im^p$ by $(i+1)1^q$.

Case (II). $d_{n+1}\cdots d_{n+q} = (i+1)1^{q-1}$. Then by (3.7) it follows that $d_{n+q+1} = 1$ or 2 .

- If $d_{n+q+1} = 1$, then we have a substitution by replacing

$$d_{n+1} \cdots d_{n+q+1} = (i+1)1^q$$

by im^p .

- If $d_{n+q+1} = 2$, then by (3.5) and (3.7) it yields that

$$\pi(d_{n+q+1}d_{n+q+2} \cdots) \in f_1(I) \cap f_2(I) = f_{1m^u}(I) = f_{21^v}(I).$$

So, by Lemma 2.2 it follows that $d_{n+q+1} \cdots d_{n+q+v} = 21^{v-1}$. Furthermore, $d_{n+q+v+1} = 1$ or 2 . Note by the assumption that (d_i) does not end with $(21^{v-1})^\infty$. Then by iteration it follows that there exists $N \geq n + v + 1$ such that

$$d_{N+1} \cdots d_{N+v+1} = 21^v.$$

Hence, we also have a substitution by considering $21^v \sim 1m^u$.

By Cases (I)–(II) and (3.6) it follows that there exist infinitely many independent substitutions in (d_i) . This implies that x has a continuum of codings, leading to a contradiction with $x \in \mathcal{U}_{\aleph_0}(E)$. \square

Proof of Theorems 1.1 and 1.2 (i) \Leftrightarrow (iv). The equivalences of (i) and (iv) follows directly by Lemmas 3.2 and 3.3. \square

4. UNCOUNTABLE CODINGS

In this section we will consider the set $\mathcal{U}_{2^{\aleph_0}}(E)$ which contains all points having a continuum of codings, and prove Theorem 1.5. First let us recall that the system $(E, \{f_i\}_{i=1}^m)$ coming from the collection \mathcal{E} described in section 1. We say that i is an admissible initial code of $x \in E$ if $x \in f_i(I)$. Then each $x \in E$ has at least one admissible initial code and at most two admissible initial codes.

As we know, when $f_i(I) \cap f_{i+1}(I) \neq \emptyset$ there exists a unique positive integer pair $(u(i), v(i))$ such that $f_{im^{u(i)}}(I) = f_{(i+1)1^{v(i)}}(I)$. Let $u = \max_i u(i)$ and $v = \max_i v(i)$. Let

$$\mathcal{P} = \bigcup_{i=1}^m \{f_i(a), f_i(b)\} \cup \bigcup_{\ell=1}^v \{f_{1^\ell}(b)\} \cup \bigcup_{\ell=1}^u \{f_{m^\ell}(a)\}.$$

The first set $\bigcup_{i=1}^m \{f_i(a), f_i(b)\}$ consists of the endpoints of the fundamental intervals $f_i(I)$, $1 \leq i \leq m$. Now we list the elements of \mathcal{P} in

increasing order and write

$$\mathcal{P} = \{s_j : 1 \leq j \leq \gamma\},$$

where $\gamma = \#\mathcal{P} = 2m + u + v - 2$. Note that $f_{1^2}(b) < f_2(a)$ and $f_{m-1}(b) < f_{m^2}(a)$. Then the first v members and the last u members of \mathcal{P} are

$$s_1 = f_1(a) = a < s_2 = f_{1^v}(b) < s_3 = f_{1^{v-1}}(b) < \cdots < s_v = f_{1^2}(b)$$

and

$$s_{\gamma-u+1} = f_{m^2}(a) < s_{\gamma-u+2} = f_{m^3}(a) < \cdots < s_{\gamma-1} = f_{m^u}(a) < s_\gamma = f_m(b) = b.$$

For two consecutive members s_i and s_{i+1} of \mathcal{P} , we call them an *admissible pair* if there exists a j such that

$$(4.1) \quad s_i, s_{i+1} \in f_j(I).$$

Let

$$(4.2) \quad \mathcal{Q} = \{1 \leq i < \gamma : \{s_i, s_{i+1}\} \text{ is an admissible pair}\}.$$

For an admissible pair $\{s_i, s_{i+1}\}$, there exist at most two j 's satisfying (4.1) and we denote by $\alpha(i)$ the smaller j . One can verify that $f_{\alpha(i)}^{-1}(s_i), f_{\alpha(i)}^{-1}(s_{i+1}) \in \mathcal{P}$. For $s, t \in \mathcal{P}$ with $s < t$ let

$$\mathcal{V}[s, t] = \{\{s_j, s_{j+1}\} : j \in \mathcal{Q} \text{ and } [s_j, s_{j+1}] \subseteq [s, t]\}$$

For an admissible pair $\{s_i, s_{i+1}\}$ let

$$\mathcal{A}\{s_i, s_{i+1}\} = \mathcal{V}[f_{\alpha(i)}^{-1}(s_i), f_{\alpha(i)}^{-1}(s_{i+1})] \text{ and } [s_i, s_{i+1}]_E = [s_i, s_{i+1}] \cap E.$$

The following properties can be verified:

- (I) We have $E = \bigcup_{i \in \mathcal{Q}} [s_i, s_{i+1}]_E$.
- (II) The compact sets $[s_i, s_{i+1}]_E, i \in \mathcal{Q}$ obey a graph-directed structure:

$$[s_i, s_{i+1}]_E = \bigcup_{\{s_j, s_{j+1}\} \in \mathcal{A}\{s_i, s_{i+1}\}} f_{\alpha(i)}([s_j, s_{j+1}]_E).$$

In addition, it is clear that the above graph-directed structure satisfies the open set condition with respect to the open sets $\{(s_i, s_{i+1}) : i \in \mathcal{Q}\}$.

We remark that the above properties actually give a way to calculate $\dim_H E$. Now we construct a directed graph \mathcal{G} by taking $\mathcal{V} = \{\{s_j, s_{j+1}\} : i \in \mathcal{Q}\}$ as the vertex set. For two vertices $\{s_i, s_{i+1}\}$ and

$\{s_j, s_{j+1}\}$ we connect a directed edge from $\{s_i, s_{i+1}\}$ to $\{s_j, s_{j+1}\}$, denoted by $\{s_i, s_{i+1}\} \rightarrow \{s_j, s_{j+1}\}$, if $[s_j, s_{j+1}] \in \mathcal{A}\{s_i, s_{i+1}\}$. We say vertex $\{s_i, s_{i+1}\}$ can be connected to vertex $\{s_j, s_{j+1}\}$ by edges, denoted by $\{s_i, s_{i+1}\} \Rightarrow \{s_j, s_{j+1}\}$, if either $\{s_i, s_{i+1}\} \rightarrow \{s_j, s_{j+1}\}$ or there exist vertices B_1, \dots, B_n such that

$$\{s_i, s_{i+1}\} \rightarrow B_1 \rightarrow \dots \rightarrow B_n \rightarrow \{s_j, s_{j+1}\}.$$

Lemma 4.1. *The directed graph \mathcal{G} is strongly connected.*

Proof. Arbitrarily fix two vertices $\{s_i, s_{i+1}\}$ and $\{s_j, s_{j+1}\}$. We need to show $\{s_i, s_{i+1}\} \Rightarrow \{s_j, s_{j+1}\}$. The proof is divided into three cases.

Case I. $\{s_i, s_{i+1}\} \in \{\{s_1, s_2\}, \{s_{\gamma-1}, s_\gamma\}\}$.

Without loss of generality we assume that $\{s_i, s_{i+1}\} = \{s_1, s_2\} = \{f_1(a), f_{1^v}(b)\}$. The case $\{s_i, s_{i+1}\} = \{s_{\gamma-1}, s_\gamma\} = \{f_{m^u}(a), f_m(b)\}$ can be dealt with in the same way. By definition the following connections hold: $\{s_1, s_2\} \rightarrow \{s_1, s_2\}$, and

$$\{s_1, s_2\} \rightarrow \{s_2, s_3\} \rightarrow \dots \rightarrow \{s_{v-1}, s_v\} = \{f_{1^3}(b), f_{1^2}(b)\}.$$

Note that

$$\mathcal{A}\{f_{1^3}(b), f_{1^2}(b)\} = \begin{cases} \{\{f_{1^2}(b), f_1(b)\}\} & \text{if } f_1(I) \cap f_2(I) = \emptyset \\ \{\{f_{1^2}(b), f_2(a)\}, \{f_2(a), f_1(b)\}\} & \text{if } f_1(I) \cap f_2(I) \neq \emptyset. \end{cases}$$

When $f_1(I) \cap f_2(I) = \emptyset$ we have

$$\{f_{1^3}(b), f_{1^2}(b)\} \rightarrow \{f_{1^2}(b), f_1(b)\} \rightarrow \{s, t\} \text{ for all } \{s, t\} \in \mathcal{V}[f_1(b), b].$$

Thus the result is correct. When $f_1(I) \cap f_2(I) \neq \emptyset$ we have

$$\{f_{1^3}(b), f_{1^2}(b)\} \rightarrow \{f_{1^2}(b), f_2(a)\} \rightarrow \{s, t\} \text{ for all } \{s, t\} \in \mathcal{V}[f_1(b), f_{m^u(1)}(a)].$$

and

$$\{f_{1^3}(b), f_{1^2}(b)\} \rightarrow \{f_2(a), f_1(b)\} \rightarrow \{s, t\} \text{ for all } \{s, t\} \in \mathcal{V}[f_{m^u(1)}(a), b].$$

This implies the result is correct.

Case II. $\{s_i, s_{i+1}\} \in \mathcal{V}[a, f_1(b)] \cup \mathcal{V}[f_m(a), b]$.

Without loss of generality we assume that $\{s_i, s_{i+1}\} \in \mathcal{V}[a, f_1(b)]$.

The case $\{s_i, s_{i+1}\} \in \mathcal{V}[f_m(a), b]$ can be dealt with in the same way.

When $f_1(I) \cap f_2(I) = \emptyset$, the fact $\{s_i, s_{i+1}\} \Rightarrow \{f_{m^u}(a), b\}$ can be derived directly from the discussion in case I. So the result is correct.

For the case that $f_1(I) \cap f_2(I) \neq \emptyset$ it suffices to show $\{f_{1^2}(b), f_2(a)\} \Rightarrow \{f_{m^u}(a), b\}$. Note that

$$\begin{cases} \{f_1(b), f_2(b)\} \in \mathcal{V}[f_1(b), f_{m^{u(1)}}(a)] & \text{when } f_2(I) \cap f_3(I) = \emptyset \\ \{f_3(a), f_2(b)\} \in \mathcal{V}[f_1(b), f_{m^{u(1)}}(a)] & \text{when } f_2(I) \cap f_3(I) \neq \emptyset. \end{cases}$$

Thus either

$$\{f_{1^2}(b), f_2(a)\} \rightarrow \{f_1(b), f_2(b)\} \rightarrow \{s, t\} \text{ for all } \{s, t\} \in \mathcal{V}[f_{1^{v(1)}}(b), b]$$

or

$$\{f_{1^2}(b), f_2(a)\} \rightarrow \{f_3(a), f_2(b)\} \rightarrow \{s, t\} \text{ for all } \{s, t\} \in \mathcal{V}[f_{m^{u(2)}}(a), b].$$

Thus we have $\{f_{1^2}(b), f_2(a)\} \Rightarrow \{f_{m^u}(a), b\}$.

Case III. $\{s_i, s_{i+1}\} \in \mathcal{V}[f_1(b), f_m(a)]$.

For this case the admissible pair $\{s_i, s_{i+1}\}$ may occur as the following five forms:

(i) $\{s_i, s_{i+1}\} = \{f_k(a), f_{k+1}(a)\}$. Then we have

$$\{s_i, s_{i+1}\} = \{f_k(a), f_{k+1}(a)\} \rightarrow \{s, t\} \in \mathcal{V}[a, f_{m^{u(k)}}(a)].$$

This reduces to Case II.

(ii) $\{s_i, s_{i+1}\} = \{f_{k+1}(a), f_k(b)\}$. Then we have

$$\{s_i, s_{i+1}\} = \{f_{k+1}(a), f_k(b)\} \rightarrow \{s, t\} \in \mathcal{V}[f_{m^{u(k)}}(a), b].$$

This reduces to Case II.

(iii) $\{s_i, s_{i+1}\} = \{f_k(b), f_{k+1}(b)\}$. Then we have

$$\{s_i, s_{i+1}\} = \{f_k(b), f_{k+1}(b)\} \rightarrow \{s, t\} \in \mathcal{V}[f_{1^{v(k)}}(b), b].$$

This reduces to Case II.

(iv) $\{s_i, s_{i+1}\} = \{f_k(a), f_k(b)\}$. Then we have

$$\{s_i, s_{i+1}\} = \{f_k(a), f_k(b)\} \rightarrow \{s, t\} \in \mathcal{V}[a, b].$$

This reduces to Case II.

(v) $\{s_i, s_{i+1}\} = \{f_k(b), f_{k+2}(a)\}$. Then we have

$$\{s_i, s_{i+1}\} = \{f_k(b), f_{k+2}(a)\} \rightarrow \{s, t\} \in \mathcal{V}[f_{1^{v(k)}}(b), f_{m^{u(k+1)}}(a)].$$

Note that $[f_1(b), f_m(a)] \subseteq [f_{1^{v(k)}}(b), f_{m^{u(k+1)}}(a)]$. Then there exists an admissible pair $\{s, t\} \in \mathcal{V}[f_1(b), f_m(a)]$ belonging to one of types (i)–(iv), and $\{s_i, s_{i+1}\} \rightarrow \{s, t\}$. Thus the result follows by (i)–(iv).

The result now is proved by the above three cases. \square

In the following lemma we show that the Hausdorff dimension of $\mathcal{U}_1(E)$ is strictly smaller than $\dim_H E$.

Lemma 4.2. *Let $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$. Then $\dim_H \mathcal{U}_1(E) < \dim_H E$. Furthermore,*

$$0 < \mathcal{H}^{\dim_H E}(E) < \infty.$$

Proof. By (II) and Lemma 4.1 it follows that E is a strongly connected graph-directed set satisfying the open set condition. Then by [13, Theorem 1.3] we obtain

$$0 < \mathcal{H}^{\dim_H E}(E) < \infty.$$

Let \mathcal{Q}^* be the subset of \mathcal{Q} defined in (4.2) by deleting those j for which $\{s_j, s_{j+1}\} = \{f_{\ell+1}(a), f_\ell(b)\}$ for some ℓ . For this \mathcal{Q}^* , one can get a graph-directed set E^* for which $\dim_H E^* < \dim_H E$. Moreover, $U_1(E)$ is a subset of E^* . Hence $\dim_H U_1(E) < \dim_H E$. \square

Proof of Theorem 1.5. Note that

$$E = \mathcal{U}_{2^{\aleph_0}}(E) \cup \mathcal{U}_{\aleph_0}(E) \cup \bigcup_{k=1}^{\infty} \mathcal{U}_k(E).$$

Furthermore, by Theorems 1.1–1.2 and Lemma 4.2 it follows that

$$\dim_H \left(\mathcal{U}_{\aleph_0}(E) \cup \bigcup_{k=1}^{\infty} \mathcal{U}_k(E) \right) = \dim_H \mathcal{U}_1 < \dim_H E.$$

Therefore, by Lemma 4.2 we have

$$\dim_H \mathcal{U}_{2^{\aleph_0}}(E) = \dim_H E \quad \text{and} \quad 0 < \mathcal{H}^{\dim_H E}(\mathcal{U}_{2^{\aleph_0}}(E)) < \infty.$$

\square

Finally we give one example which can assist us in a better understanding of our proof.

Example 4.3. Let E be the self-similar set generated by $f_1(x) = \lambda x$, $f_2(x) = \lambda x + \lambda - \lambda^2$, $f_3(x) = \lambda x + 1 - 2\lambda + \lambda^2$ and $f_4(x) = \lambda x + 1 - \lambda$

where $0 < \lambda < \frac{1}{4}$. Then $I = [0, 1]$ and one can check that $(E, \{f_i\}_{i=1}^4) \in \mathcal{E}$. In particular, we have

$$\begin{cases} f_1(I) \cap f_2(I) = f_{14}(I) = f_{21}(I) \\ f_3(I) \cap f_4(I) = f_{34}(I) = f_{41}(I) \\ f_2(I) \cap f_3(I) = \emptyset. \end{cases}$$

Hence, by Theorem 1.1 it follows that $|U_{\aleph_0}(E)| = \aleph_0$, and

$$\dim_H U_k(E) = \dim_H U_1(E) = \frac{\log 3}{-\log \lambda}.$$

The calculation of $\dim_H U_1(E)$ is due to Theorem 1.5 and [3, Theorem 2.21].

In fact we have

$$\mathcal{P} = \{0, \lambda - \lambda^2, \lambda, 2\lambda - \lambda^2, 1 - 2\lambda + \lambda^2, 1 - \lambda, 1 - \lambda + \lambda^2, 1\}.$$

Let $A_1 = [0, \lambda - \lambda^2] \cap E$, $A_2 = [\lambda - \lambda^2, \lambda] \cap E$, $A_3 = [\lambda, 2\lambda - \lambda^2] \cap E$, $A_4 = [1 - 2\lambda + \lambda^2, 1 - \lambda] \cap E$, $A_5 = [1 - \lambda, 1 - \lambda + \lambda^2] \cap E$, $A_6 = [1 - \lambda + \lambda^2, 1] \cap E$. It is easy to check that

$$\begin{cases} A_1 = f_1(A_1) \cup f_1(A_2) \cup f_1(A_3) \cup f_1(A_4) \\ A_2 = f_1(A_5) \cup f_1(A_6) \\ A_3 = f_2(A_3) \cup f_2(A_4) \cup f_2(A_5) \cup f_2(A_6) \\ A_4 = f_3(A_1) \cup f_3(A_2) \cup f_3(A_3) \cup f_3(A_4) \\ A_5 = f_3(A_5) \cup f_3(A_6) \\ A_6 = f_4(A_3) \cup f_4(A_4) \cup f_4(A_5) \cup f_4(A_6) \end{cases}$$

Hence the adjacency matrix (see [3]) is

$$S = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Note that A_2, A_5 are the switch or overlap regions. Therefore we can define an adjacency matrix for the univoque set, see [3], i.e.

$$S' = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

Using these two matrices, we can explicitly calculate the Hausdorff dimension of E as well as $U_1(E)$, see [3, Theorems 2.11, 2.21].

5. FURTHER REMARKS

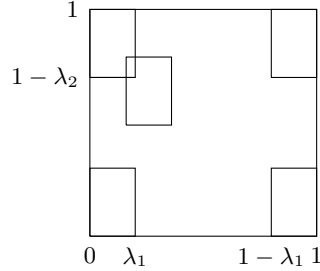
We finish this paper by giving some remarks.

- Similar idea can also be implemented in higher dimensions.
- For the self-affine setting, our idea is still working. We shall make use of the following example to illustrate this point.

Example 5.1. Let K be the self-affine set generated by the IFS

$$\begin{aligned} &\{f_1(x, y) = (\lambda_1 x, \lambda_2 y), f_2(x, y) = (\lambda_1 x + 1 - \lambda_1, \lambda_2 y), \\ &f_3(x, y) = (\lambda_1 x + 1 - \lambda_1, \lambda_2 y + 1 - \lambda_2), f_4(x, y) = (\lambda_1 x, \lambda_2 y + 1 - \lambda_2), \\ &f_5(x, y) = (\lambda_1 x + \lambda_1(1 - \lambda_1), \lambda_2 y + (1 - \lambda_2)^2)\}. \end{aligned}$$

where $0 < \lambda_1, \lambda_2 < \frac{3 - \sqrt{5}}{2}$. Let $I = [0, 1]^2$, then the first iteration of $\{f_i(I)\}_{i=1}^5$ is the following figure



For this example, note that $f_{42} = f_{54}$, and $f_i(I) \cap f_j(I) = \emptyset, 1 \leq i \leq 3, j \neq i$. Using similar ideas of Lemmas 2.3, 2.4, 2.5, we can show that

$$\dim_H(U_k(K)) = \dim_H(U_1(K))$$

for any finite $k \geq 2$.

Finally we remark that it would be interesting to consider multiple codings in the case of β -expansions. We will investigate in a separated paper.

ACKNOWLEDGEMENTS

The second author was granted by the China Scholarship Council. The third author was supported by NSFC No. 11401516 and Jiangsu Province Natural Science Foundation for the Youth no BK20130433. The forth author was supported by NSFC No. 11271137, 11571144 and Science and Technology Commission of Shanghai Municipality (STCSM), grant No. 13dz2260400.

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DEPARTMENT OF MATHEMATICS, UTRECHT UNIVERSITY, FAC WISKUNDE EN INFORMATICA AND MRI, BUDAPESTLAAN 6, P.O. Box 80.000, 3508 TA UTRECHT, THE NETHERLANDS

E-mail address: k.dajani1@uu.nl

DEPARTMENT OF MATHEMATICS, UTRECHT UNIVERSITY, FAC WISKUNDE EN INFORMATICA AND MRI, BUDAPESTLAAN 6, P.O. Box 80.000, 3508 TA UTRECHT, THE NETHERLANDS

E-mail address: K.Jiang1@uu.nl

SCHOOL OF MATHEMATICAL SCIENCE, YANGZHOU UNIVERSITY, YANGZHOU, JIANGSU 225002, PEOPLE’S REPUBLIC OF CHINA

E-mail address: derongkong@126.com

DEPARTMENT OF MATHEMATICS, SHANGHAI KEY LABORATORY OF PMMP, EAST CHINA NORMAL UNIVERSITY, SHANGHAI 200062, PEOPLE’S REPUBLIC OF CHINA

E-mail address: wxli@math.ecnu.edu.cn